SYDNEY TECHNICAL HIGH SCHOOL

Mathematics Extension 2

HSC ASSESSMENT TASK
JUNE 2006

General Instructions

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions

NAME :________________________

QUESTION 1	QUESTION 2	QUESTION 3	TOTAL

QUESTION ONE (16 marks) Marks

a) Find $\int \cos^3 x \, dx$ 3

b) Use partial fractions to find $\int \frac{24}{x^2 + 4x - 12} \, dx$ 3

c) Find $\int \frac{24}{x^2 + 6x + 18} \, dx$ 2

d) Evaluate $\int_0^{\frac{\pi}{6}} x \cos x \, dx$ 3

e) Solve the equation $x^3 - 5x^2 + 11x - 15 = 0$ 2

given that $x = 1 - 2i$ is a solution.

f) For the polynomial $P(x) = x^3 + (k+3)x^2 + (2k+7)x + (k+5)$ 3

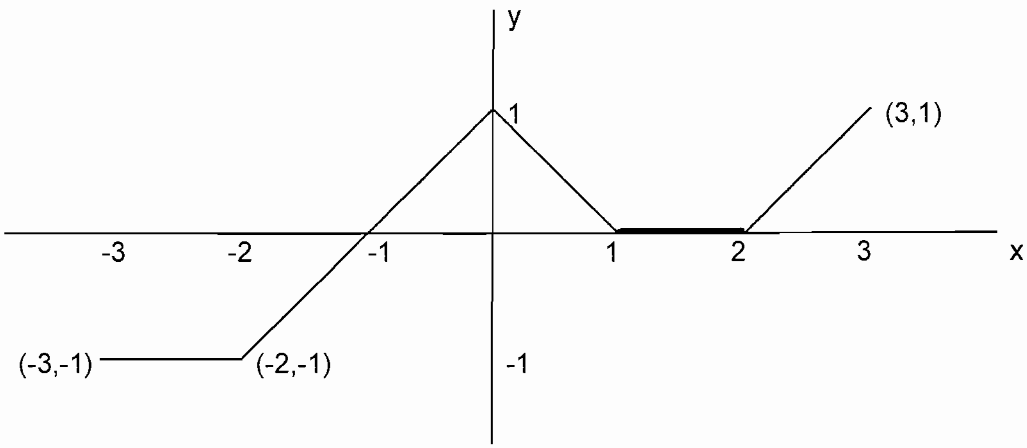
where k is real, find the possible values of k given that $x = -1$

is the only real root of $P(x) = 0$.

QUESTION TWO (17 marks) (Start a new page) Marks

a) Find $\int \frac{x+1}{\sqrt{x-1}}\,dx$ 2

b) The diagram below is a sketch of the function $y = f(x)$ defined for $-3 \le x \le 3$



On separate diagrams sketch

i) $y = \frac{1}{f(x)}$ 2

ii) $y = \log_e f(x)$ 2

iii) $y^2 = f(x)$ 2

c) i) Use the identity $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ 3

to solve the equation $16x^4 - 16x^2 + 1 = 0$

ii) Use the above solutions to show that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$ 3

Justify your answer

iii) By solving the equation $16x^4 - 16x^2 + 1 = 0$ 3

using a different method from above,

find the exact value of $\cos\frac{\pi}{12}$.

Justify your answer.

QUESTION THREE (17 marks) (Start a new page) Marks

a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ using the substitution $t = \tan\frac{x}{2}$ 3

b) Solve $4x^3 + 4x^2 - 7x + 2 = 0$ 3

given that two of the roots are equal.

c) The equation $x^3 + px + q = 0$ (p, q are real) has roots α, β and δ.

i) Find the polynomial equation with roots 3α, 3β and 3δ. 2

ii) Show that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\delta^2}{q}$ 1

iii) Find a cubic polynomial equation with roots 2

$\frac{1}{\alpha} + \frac{1}{\beta}$, $\frac{1}{\beta} + \frac{1}{\delta}$ and $\frac{1}{\alpha} + \frac{1}{\delta}$

d) i) Show that $(1-x)^{n-1} - (1-x)(1-x)^{n-1} = x(1-x)^{n-1}$ 1

ii) If $I_n = \int_0^1 \sqrt{x}\,(1-x)^n\,dx$ where n is a non negative integer

Show that $I_n = \frac{2n}{2n+3} I_{n-1}$ 3

iii) Evaluate $\int_0^1 \sqrt{x}\,(1-x)^3\,dx$ 2

SOLUTIONS 2006

Q1. a. $\int \cos^3 x \, dx$

$= \int \cos x \cos^2 x \, dx$

$= \int \cos x \left(1 - \sin^2 x\right) dx$

let $u = \sin x$

$du = \cos x \, dx$

$= \int 1 - u^2 \, du$

$= u - \frac{1}{3}u^3$

$= \sin x - \frac{1}{3}\sin^3 x + c$

b. $\dfrac{24}{x^2+4x-12} = \dfrac{A}{x+6} + \dfrac{B}{x-2}$

$\therefore \quad 24 = A(x-2) + B(x+6)$

when $x = -6$

$24 = -8A$

$A = -3$

when $x = 2$

$24 = 8B$

$B = 3$

$\therefore \quad \int \dfrac{24}{x^2+4x-12} \, dx$

$= \int \dfrac{3}{x-2} - \dfrac{3}{x+6} \, dx$

$= 3\ln(x-2) - 3\ln(x+6) + c$

$= 3\ln\left(\dfrac{x-2}{x+6}\right) + c$

c. $\displaystyle\int \frac{24}{x^2+6x+18}\,dx$

$$= \int \frac{24}{(x+3)^2+9}\,dx$$

$$= \frac{24}{3}\tan^{-1}\frac{x+3}{3} + c$$

$$= 8\tan^{-1}\left(\frac{x+3}{3}\right) + c$$

d. $\displaystyle\int_0^{\frac{\pi}{6}} x\cos x\,dx$

$$= x\sin x\Big]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x\,dx$$

$$= \frac{\pi}{12} + \Big[\cos x\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

e. roots $1-2i,\ 1+2i,\ \alpha$

sum of roots $= 5$

$\therefore\ 1-2i+1+2i+\alpha = 5$

$\alpha = 3$

$\therefore$ Solutions are $1-2i,\ 1+2i,\ 3$

f.

$$\begin{array}{r|l}
 & x^2 + (k+2)x + (k+5) \\
x+1 & x^3 + (k+3)x^2 + (2k+7)x + (k+5) \\
 & \underline{x^3 + x^2} \\
 & (k+2)x^2 + (2k+7)x + (k+5) \\
 & \underline{(k+2)x^2 + (k+2)x} \\
 & (k+5)x + (k+5) \\
 & \underline{(k+5)x + (k+5)} \\
 & -
\end{array}$$

$\therefore\ P(x) = (x+1)\left(x^2 + (k+2)x + (k+5)\right)$

quadratic has no real roots

$\therefore \Delta < 0$

$\therefore \quad (k+2)^2 - 4(1)(k+5) < 0$

$k^2 - 16 < 0$

$\underline{-4 < k < 4}$

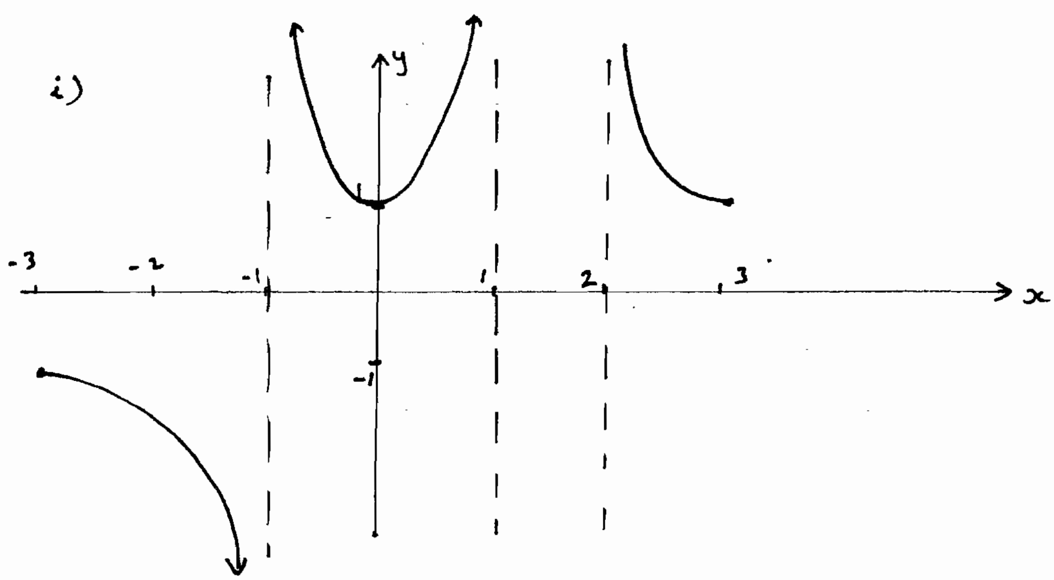
Q2. a. $\displaystyle\int \frac{x+1}{\sqrt{x-1}}\,dx$

$= \displaystyle\int \frac{x-1}{\sqrt{x-1}} + \frac{2}{\sqrt{x-1}}\,dx$

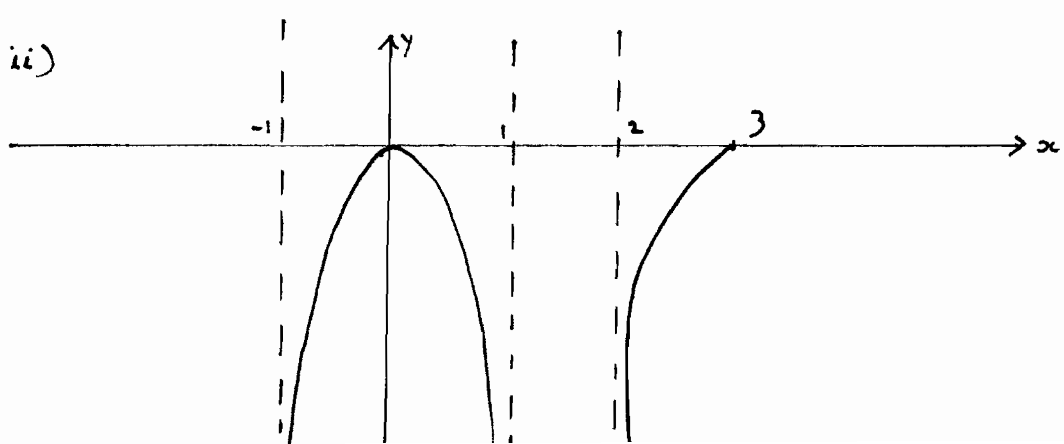
$= \displaystyle\int (x-1)^{\frac{1}{2}} + 2(x-1)^{-\frac{1}{2}}\,dx$

$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 4\sqrt{x-1} + c$

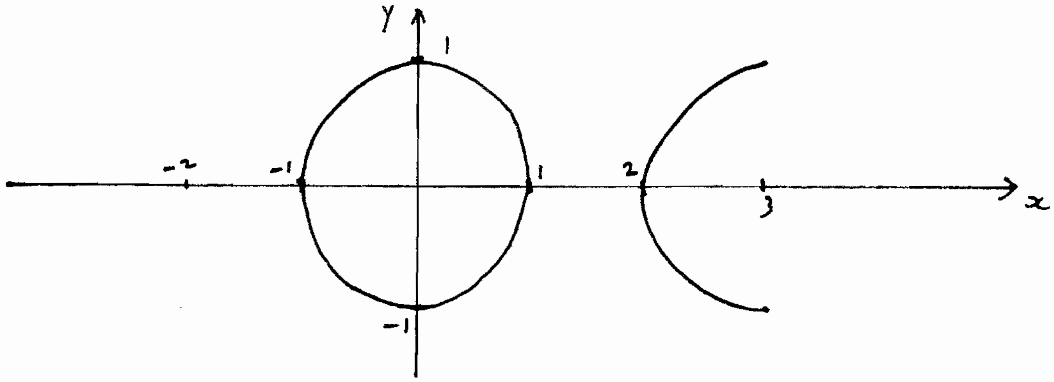
b. i)



ii)



iii)



c i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

let $\cos\theta = x$

$\therefore \quad 16x^4 - 16x^2 + 1 = 0$

$16\cos^4\theta - 16\cos^2 x + 1 = 0$

$2(8\cos^4\theta - 8\cos^2 x + 1) - 1 = 0$

$2\cos 4\theta - 1 = 0$

$\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

$\therefore \quad x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$ (or equivalent)

ii) product of roots $= \frac{1}{16}$

$\therefore \quad \cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{1}{16}$

but $\cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}$

$\cos\frac{11\pi}{12} = -\cos\frac{\pi}{12}$

$\therefore \quad \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}$

$\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \pm\frac{1}{4}$ (but all angles in 1st quadrant)

$\therefore \cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$

iii) $$16x^4 - 16x^2 + 1 = 0$$

let $u = x^2$

$$16u^2 - 16u + 1 = 0$$

$$\therefore \quad u = \frac{16 \pm \sqrt{16^2 - 4 \times 16 \times 1}}{32}$$

$$= \frac{1}{2} \pm \frac{1}{4}\sqrt{3}$$

$$\therefore \quad x^2 = \frac{1}{2} \pm \frac{1}{4}\sqrt{3}$$

$$\therefore \quad x = \pm\sqrt{\frac{1}{2} \pm \frac{1}{4}\sqrt{3}}$$

$$\therefore \quad \cos\frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}}$$ (largest all positive)

Q3. a. $$\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x}$$

$t = \tan\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$

$$= \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2\,dt}{t^2 + 3}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\frac{t}{\sqrt{3}}\Big]_0^1$$

$$= \frac{\pi}{3\sqrt{3}}$$

b. $$P(x) = 4x^3 + 4x^2 - 7x + 2$$

$$P'(x) = 12x^2 + 8x - 7$$

double root is a root of $P'(x) = 0$

$$12x^2 + 8x - 7 = 0$$

$$(6x + 7)(2x - 1) = 0$$

$P(\frac{1}{2}) = 0$

$\therefore$ $x = \frac{1}{2}$ is double root

$\therefore$ roots are $\frac{1}{2}, \frac{1}{2}, \alpha$

sum of roots $= -1$

$$\frac{1}{2} + \frac{1}{2} + \alpha = -1$$

$$\alpha = -2$$

$\therefore$ Solutions $\frac{1}{2}, \frac{1}{2}, -2$

c. i) $P(\frac{x}{3}) = 0$

$$\left(\frac{x}{3}\right)^3 + p\left(\frac{x}{3}\right) + q = 0$$

$$\frac{x^3}{27} + \frac{px}{3} + q = 0$$

$$x^3 + 9px + 27q = 0$$

ii) $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$= \frac{\delta(\alpha+\beta+\delta-\delta)}{\alpha\beta\delta}$$

$\alpha+\beta+\delta = 0$

$\alpha\beta\delta = -9$

$$= \frac{-\delta^2}{-9}$$

$$= \frac{\delta^2}{9}$$

iii) $\frac{1}{\alpha} + \frac{1}{\beta}, \quad \frac{1}{\beta} + \frac{1}{\delta}, \quad \frac{1}{\alpha} + \frac{1}{\delta}$

is equivalent to (from above)

$$\frac{\delta^2}{9}, \quad \frac{\alpha^2}{9}, \quad \frac{\beta^2}{9}$$

$\therefore$ required polynomial is

$$P\left(\sqrt{qx}\right) = 0$$

$$\left(\sqrt{qx}\right)^3 + p\sqrt{qx} + q = 0$$

$$\left(\sqrt{qx}\right)^3 + p\sqrt{qx} = -q$$

$$\left(\left(\sqrt{qx}\right)^3 + p\sqrt{qx}\right)^2 = q^2$$

$$q^3x^3 + 2pq^2x^2 + p^2qx - q^2 = 0$$

d. i)

$$\begin{aligned} LHS &= (1-x)^{n-1} - (1-x)(1-x)^{n-1} \\ &= (1-x)^{n-1}\left[1 - (1-x)\right] \\ &= x(1-x)^{n-1} \\ &= RHS \end{aligned}$$

ii)

$$I_n = \int_0^1 \sqrt{x}\,(1-x)^n\,dx \qquad u = (1-x)^n \qquad v = \tfrac{2}{3}x^{\frac{3}{2}}$$

$$u' = -n(1-x)^{n-1} \qquad v' = x^{\frac{1}{2}}$$

$$= \cancel{\tfrac{2}{3}x^{\frac{3}{2}}(1-x)^n\Big]_0^1} + \tfrac{2}{3}n\int_0^1 x^{\frac{3}{2}}(1-x)^{n-1}\,dx$$

0

$$= \tfrac{2}{3}n\int_0^1 \sqrt{x}\,x\,(1-x)^{n-1}\,dx$$

$$= -\tfrac{2}{3}n\int_0^1 \sqrt{x}\,(1-x)(1-x)^{n-1}\,dx + \tfrac{2}{3}n\int_0^1 \sqrt{x}\,(1-x)^{n-1}\,dx$$

(using part i))

$$= -\tfrac{2}{3}n\,I_n + \tfrac{2}{3}n\,I_{n-1}$$

$$\therefore \quad I_n = -\frac{2}{3} n I_n + \frac{2}{3} n I_{n-1}$$

$$\left(\frac{2}{3}n+1\right) I_n = \frac{2}{3} n I_{n-1}$$

$$\frac{2n+3}{3} I_n = \frac{2n}{3} I_{n-1}$$

$$I_n = \frac{2n}{2n+3} I_{n-1}$$

iii) $$I_3 = \int_0^1 \sqrt{x}\,(1-x)^3\,dx$$

$$= \frac{6}{9} I_2$$

$$= \frac{6}{9}\left[\frac{4}{7} I_1\right]$$

$$= \frac{6}{9}\cdot\frac{4}{7}\left[\frac{2}{5} I_0\right]$$

$$= \frac{6}{9}\cdot\frac{4}{7}\cdot\frac{2}{5}\int_0^1 \sqrt{x}\,dx$$

$$= \frac{6}{9}\cdot\frac{4}{7}\cdot\frac{2}{5}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= \frac{6}{9}\cdot\frac{4}{7}\cdot\frac{2}{5}\cdot\frac{2}{3}$$

$$= \frac{32}{315}$$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$